1 Numerically Classifying Exceptional Sequences

An exceptional sequence in a triangulated category $\mathcal{T}$ is a sequence of objects satisfying homological conditions allowing us to give an explicit description of many aspects of the structure of $\mathcal{T}$. The first study of these objects was done for derived categories of Del Pezzo surfaces in the Rudakov seminar ([Rud90]) in 1990, where it was proven in particular that such sequences always exist. Moreover, in [KO95], Kuleshov and Orlov showed that the braid group naturally acts naturally on the set of such sequences in a transitive manner. Our aim is to understand these phenomena in the context of noncommutative algebraic geometry. A guiding question is:

Question 1. Which triangulated categories admit an exceptional sequence and what conditions are necessary to ensure the braid group acts transitively?

A fruitful step in this program is to analyze the problem numerically: any exceptional sequence in $\mathcal{T}$ gives rise to an exceptional basis for the corresponding Grothendieck group Λ and the braid group now acts linearly on the set of these exceptional bases. When Λ is the Grothendieck group of a smooth projective surface, the Serre functor induces a *Serre automorphism* s on Λ characterized by $\langle v, sw\rangle = \langle w, v\rangle$. Moreover, it is well known that s acts unipotently on Λ. In the case where $\dim(\Lambda) = 3$ the unipotency condition translates into the famous Markov relation

$$x^2 + y^2 + z^2 = xyz.$$

among the three nonzero entries in the Gram matrix. Markov proved in [MAr79] that any positive solution to this equation can be obtained from the solution $(3, 3, 3)$ in a specific way. This implies that lattice Λ of dimension 3 where s is unipotent must be isomorphic to $K(\mathbb{P}^2)$ and that the braid group action is transitive in this case, yielding a numerical answer to Question 1 for exceptional sequences of length 3.
Performing a similar argument in the length 4 case presents a more involved system of 2 diophantine equations which can be viewed as higher analogues of Markov's equation. These were first considered by Bondal in [Bon04]. In order to analyze the solutions to these equations, further conditions on Λ turn out to be necessary. To this end, in [dTdVVdB16b], Van den Bergh and the author proved:

Theorem 1.1. *([dTdVVdB16b]) If X is any smooth projective surface, and $\Lambda = K(X)_{\text{num}}$, then*

$$\operatorname{rk}(s - \operatorname{Id}_X) \leq 2.$$

Combining these two conditions allowed us to retrieve much of the underlying geometry of X abstractly in a surprisingly simple way:

Theorem 1.2. *([dTdVVdB16b]) Let Λ be a lattice with a Serre automorphism s such that*

- *s is unipotent*
- $\mathrm{rk}(s - \mathrm{Id}) = 2$

Then it is possible give a definition of the codimension filtration, Picard group, intersection form, canonical class, degree, rank etc... which coincides with the usual notion if Λ is the numerical Grothendieck group of a smooth projective surface.

As a corollary, we were able to use this above *geometry on lattices* to solve the numerical version of Question 1 in the 4-dimensional case:

Corollary 1.3. *If Λ is a 4-dimensional lattice satisfying the conditions of Theorem 1.2, then Λ is isomorphic to one of the following:*

- $K(\mathbb{P}^1 \times \mathbb{P}^1)$
- *(type n): $\mathbb{Z}^4$ endowed with Gram matrix* $\begin{bmatrix} 1 & b & 2n & n \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

If $n = 0, 1$, these lattices correspond to $K(\mathbb{P}^2 \cup \{\bullet\})$ and $K(\mathbb{F}_1)$ respectively. If $n \neq 0, 1$, then Λ cannot be isomorphic to the Grothendieck group of a smooth projective surface.

As an interesting corollary, we obtain a generalization of Markov's theorem [MAr79] in dimension 4 describing all possible solutions to Bondal's Markov equation.
Moreover, based on overwhelming evidence we can also make the following:

Conjecture 1. *Assume Λ satisfies the conditions of Theorem 1.2. Then the braid group acts transitively on the set of exceptional bases*

These results can naturally be extended in a few ways, first one can look at the same classification problem for higher dimensions. Alternatively, one can look at a stacky version of this problem: if Λ is the Grothendieck group of a Deligne-Mumford stack, it is known that s acts *quasi*-unipotently instead.

Question 2. Can one find a classification similar to Corollary 1.3 for higher dimensions, or under the assumption of quasi-unipotency instead?

Unfortunately our techniques do no seem to generalize naturally to these cases and it seems like the methods from [Per13] could be of interest in tackling the next step.

2 A Noncommutative Del Pezzo Surface

Of course, the classification described in Corollary 1.3 only provides a partial solution as one still needs to show that the listed solutions indeed are of some geometric nature. In fact, we can restate the question from §1 in the 4 dimensional case as:

Question 1 in dim. 4: Does there exist a general construction which produces noncommutative surfaces with a full strong exceptional sequence whose grothendieck group is of type n (as in Corolllary 1.3)?

Although we do believe this to be the case, at present there does not seem to be any general techniques. However, the case $n = 2$ was solved by Presotto and the author in [dTdVP15].
This case distinguishes itself in the classification as it is the only noncommutative case where the degree (in the generalized sense of Theorem 1.2) is positive, earning it the title of the *noncommutative Del Pezzo case.*
In loc. cit. the theory of noncommutative $\mathbb{P}^1$-bundles was adapted from [VdB12] to prove:

Theorem 2.1. *([dTdVP15]) There exists a noncommutative scheme X, constructed as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ in the sense of [VdB12] which is smooth, locally Noetherian and of finite global dimension, together with a full strong exceptional sequence on X such that $K(X)$ is of type 2 in the terminology of 1.3.*

This space has an interesting local structure: on can cover X by open subschemes that are graded Morita equivalent to a new class of interesting noncommutative algebras. These were in turn studied by Presotto and the author in [dTdVP17]. More precisely, to a Frobenius algebra S over a general commutative base ring R, we associated an algebra $\Pi_R(S)$ as the quotient of the tensoralgebra of an $R\oplus S$-bimodule modulo some simple quadratic relations. In the R-split case, $\Pi_R(R^n)$ is the preprojective algebra of the star quiver. In loc.cit, we showed that in a suitable sense $\Pi_R(S)$ is always a deformation of this preprojective algebra, so that $\Pi_R(S)$ can be considered a *relative* version of the classical preprojective algebra. These deformations are different from the ones considered in [CBH98] . We proved that if $\mathrm{rk}(S/R) = 4$, these algebras are finite over their centers and if S and R are smooth, $\Pi_R(S)$ is of finite global dimension. The techniques used are heavily reliant on the rank 4 hypothesis and it would be a good challenge to see how general these statements are.

3 Tilting of Higher Preprojective Algebras

Another possible approach to defining a noncommutative Del Pezzo surfaces is based off the well-known fact that the total space of the canonical bundle of a Del Pezzo surface is 3-Calabi-Yau (3CY for short), so that any Del Pezzo surface can be viewed as the Proj of a Calabi-Yau 3-fold. This 3-fold is canonically derived equivalent to a noncommutative 3CY Jacobi algebra (in the sense of [Gin06]) of a quiver with potential ("*QP*") without loops or 2-cycles. An elegant proof of this fact was found by Bridgeland and Stern in [BS10] where they constructed such an algebra out of any exceptional sequence on X -which they called the rolled-up algebra. This led to the approach of defining a Del Pezzo surface as the noncommutative Proj of a 3CY Jacobi algebra. Another natural guiding question in the understanding of noncommutative Del Pezzo geometry is:

Question 3. To what extend does the algebraic structure of the rolled-up algebra Λ of an exceptional sequence on a Del Pezzo surface X describe the geometry of X, or of the noncommutative Del Pezzo surface $\mathrm{Proj}(\Lambda)$) ?

To give a concrete illustration of this question, recall that 3CY Jacobi-algebra of a QP without loops or 2-cycles Λ can be *mutated at a vertex k*. More precisely, in [IR08], Iyama and Reiten construct a tilting module which gives rise to a derived equivalent 3CY algebra $\mu_k(\Lambda)$ and it was shown in subsequent work that the combinatorics of this rule coincide with the ubiquitous Fomin-Zelevinsky mutation rule (see [BIRS11]) for a QP. Vice versa, A QP always defines a 3CY *dg-algebra* (see [KY11]), called the Ginzburg dg-algebra (see [Gin06]), and Fomin-Zelevinsky mutation of a QP without loops or 2-cycles lifts to a tilt of the corresponding Ginzburg dg algebras.
It was Bridgeland and Stern again ([BS10]) who interpreted tilting in the context of question 3 : given the rolled-up algebra of an exceptional sequence on a Del Pezzo surface, they showed that any tilt can be realized by performing the braid group action introduced in §1 on the underlying exceptional sequence and considering rolled-up algebra of the resulting sequence.
Since a rolled-up algebra never has a loop or 2-cycle, one byproduct of this result is that the quiver of such an algebra never acquires loops or 2-cycles under Fomin-Zelevinsky mutation. We call this property *non-degeneracy*.
In [dTdVVdB13], Van den Bergh and the author set out to understand the non-degeneracy and Calabi-Yau properties of the rolled-up algebra from a purely algebraic standpoint instead in order to apply these results more generally to the context of singularity theory for one. Crucial to this understanding were techniques developed by Keller in [Kel11]. There he introduced CY *completions*, a particular type of 3CY algebra, related to Iyama's higher preprojective algebras which play a crucial role in his *higher* representation theory. In loc. cit. Keller showed that over a base algebra of global dimension ≤ 2 these algebras are always quasi-isomorphic to a Ginzburg dg-algebra. Our main application of this construction was the following:

Theorem 3.1. *[dTdVVdB16a] Assume that (Q, w) is a graded QP of degree r. Let Λ be the graded Jacobi algebra. Then under some technical condition on the graded center of Λ (satisfied if $r = 1$), (Q, w) is non-degenerate.*

Moreover, the rolled-up algebra associated to an exceptional sequence in [BS10] is a higher preprojective algebra over a ring of global dimension ≤ 2

As an immediate corollary, we reobtain Bridgeland-Stern's result on non-degeneracy and the 3CY property of their algebra.
As a next step we wish to investigate the possibility of describing the relation between tilting the algebra and braid group action on the underlying sequence through the same techniques. In particular we wish to know if said relation holds not just for the rolled-up algebra, but more generally for higher preprojective algebras:

Question 4. When is the tilt of a higher preprojective algebra a higher preprojective algebra?

A guiding (counter)example in this program is the algebra $\Lambda := \mathbb{C}[[x_1, x_2, x_3]]\#\mathbb{Z}/5\mathbb{Z}$ where $\mathbb{Z}/5\mathbb{Z}$ acts with weights $(2, 2, 1)/5$. This algebra is derived equivalent to the minimal resolution of the quotient singularity $\mathbb{C}[[x_1, x_2, x_3]]/(\mathbb{Z}/5\mathbb{Z})$ by the derived McKay correspondence ([BKR01]). The Theorem 3.1 immediately implies that it is non-degenerate in the above sense -answering a question originally raised by Iyama-Reiten in [IR08]. It does not share some other elementary properties of Del Pezzo surfaces however: question 3 has a negative answer in this case. Moreover, there are non-strong exceptional sequence in $D^b(\Lambda)$.

4 Models of Singularity Categories

As mentioned in §3, one purpose for generalizing the results of [BS10] comes from the fact that quivers with potential play an important role in singularity theory through the work of Keller ([Kel11]), Ginzburg ([Gin06]) and many others.
Since the work of Buchweitz in [Buc86] (or [Orl04] in a different context) the category $\mathrm{Sing}(R) = D^b(R)/K(\mathrm{proj}(R))$ (which is trivial if R is nonsingular) has become a major object of study in singularity theory.
The philosophy of noncommutative crepant resolutions ("*nccrs*") introduced in [VdB04] led us to the study what is now referred to as the *relative singularity category* (see [BK12] and others): for a complete Gorenstein isolated singularity R, we pick an nccr Λ and consider $\mathrm{Sing}(R, \Lambda) = D^b(\Lambda)/K(\mathrm{Proj}(R))$ instead. Our work in [dTdVVdB16a] gave an explicit description for this category:

Theorem 4.1. *[dTdVVdB16a]) Let* Γ *be a minimal model for the nccr* Λ*. Then there is a commutative square*

$$\begin{tikzcd}
\mathrm{Sing}(R,\Lambda) \arrow[r, "\cong"] \arrow[d] & \mathrm{Perf}(\Gamma) \arrow[d] \\
\mathrm{Sing}(R) \arrow[r, "\cong"] & \mathrm{Perf}(\Gamma)/D^b_{fd}(\Gamma)
\end{tikzcd}$$

The bottom right category $\mathcal{C}_\Gamma = \mathrm{Perf}(\Gamma)/D^b_{fd}(\Gamma)$ has a very nice interpretation through the work of [Ami09] and [Kel11]. Indeed, Amiot studied the category $\mathcal{C}_\Gamma$ in the case where Γ is the higher preprojective algebra introduced in §2. In loc cit. she showed in this case that this category has the structure of a so-called *cluster category,* a categorification of the concept of a cluster algebra. In particular this category is 2-Calabi-Yau and has a *cluster tilting object*. Following the construction explicitly allows us to specialize the result to certain cases of interest:

Corollary 4.2.
- *Let R be a complete Gorenstein isolated singularity of dimension 3. Then* $\mathrm{Sing}(R)$ *is given by the cluster category associated to the Jacobi algebra of an (explicit) quiver with potential*
- *Assume $\mathbb{Z}/\mathbb{Z}_d$ acts on $\mathbb{C}[[x_1, \ldots x_d]]$ with weights satisfying a certain technical condition. Then* $\mathrm{Sing}(R)$ *is isomorphic to the cluster category of a higher preprojective algebra*

This yields another interpretation of some of the results in [AIR11].

5 Deformations of Calabi-Yau Algebras

We explained in §3 how the tilting process of Iyama-Reiten allows us to produce families of 3CY algebras. An alternate construction comes from the celebrated work of Ginzburg-Etingoff in [GE07] which describes how to produce different families of 3CY algebras through deformation theory. In [dTdVVdB17], Van den Bergh and the author took the idea of Ginzburg-Etingoff as a basis do develop a theory of CY-deformations.

It is believed since Deligne's famous letter to Millson ([Del87]) that well-posed deformation problems should be described in terms of *Maurer-Cartan* formalism in which one constructs a deformation functor (in the sense of Schlessinger [Sch68]) by means of a dg lie algebra $\mathfrak{g}^\bullet$. The classical example being the problem of deforming an associative algebra which is described by the Gerstenhaber bracket on the shifted Hochschild cochain complex $\mathfrak{C}^\bullet(A)[1]$.

In [dTdVVdB17], we proved that Ginzburg's symmetry condition $A^D \cong_{qis} A[d]$ ([Gin06]) that define CY algebras can be described by a negative cyclic cycle in $CC^-_\bullet(A)$ satisfying a condition we called *non-degeneracy.* A d-CY algebra now becomes the data (A, μ, η) of a vector space A, a Hochschild 2-cocycle μ representing the multiplication and a non-degenerate negative cyclic d-cycle η. It is for this data that we built a deformation theory.

The homological interplay between the shifted Hochschild cocomplex $\mathfrak{C}(A)[1]$ and the negative cyclic complex $CC^-_\bullet(A)$ has been described as *noncommutative calculus* by Tamarkin and Tsygan in [TT05]. In loc. cit. they define a Lie derivative which one can use to *twist* $\mathfrak{C}(A)[1]$ by $CC^-_\bullet(A)$ using η. This produced a dg Lie algebra $\mathfrak{D}^\bullet(A,\mu,\eta)$ with the required properties, yielding

Theorem 5.1. *([dTdVVdB17]) The functor describing CY deformations of a CY-algebra is of Maurer-Cartan type for a certain dg Lie algebra.*

Subsequently, we described the graded Lie algebra $H(\mathfrak{D}^\bullet(A,\mu,\eta))$. In [CS99], Chas and Sullivan define a bracket on the string homology of a compact oriented manifold M which in turn gives a bracket on $HC^-_\bullet(SM)$, the negative cyclic homology of the loop space through the Jones isomorphism. In [Men09], Menichi generalized this idea to define a bracket on the negative cyclic homology of any algebra. We used noncommutative calculus to exhibit a quasi-isomorphism of $\mathfrak{D}^\bullet(A,\mu,\eta)$ with the negative cyclic complex and showed that the induced Lie bracket coincided with Menichi's on the level of homology:

Theorem 5.2. *([dTdVVdB17]) The dg Lie algebra governing the deformation theory of a Calabi-Yau algebra has negative cyclic homology with bracket induced by the Chas-Sullivan string topology bracket.*

General considerations in the theory of deformation functors now allow us to conclude:

Corollary 5.3. *([dTdVVdB17]) The obstruction space of the deformation problem lies in the kernel of $HC^-_{d-3}(A) \longrightarrow HC^{per}_{d-3}(A)$ (in particular 3CY algebras are unobstructed).*

The following question thus arises:

Question 5. Is it possible to compute the obstruction to Calabi-Yau deformations explicitly in higher dimensions? What are they in the case of Sklyanin algebras for example?

Finally we mention that the question raised in §3 based on the work in [BS10] has a very natural variant in this context and is currently still unresolved:

Question 3 for deformations: Is a CY deformation of a rolled up algebra in turn a rolled up algebra?

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