University of Toronto

Department of Computer & Mathematical Sciences

STAB57: an Introduction to Statistics

Week 8 Assignment

taught by Louis de Thanhoffer de Volcsey -email me

-website

-textbook

This week's list of problems is based on the material from:
Chapter 6, §3, Chapter 7, §1
You are expected to work on this list of problems prior to the upcoming tutorial.
Problems have the following tags:
: difficult, : Book exercise, : extra exercise

Terminology and Concepts to learn:

- hypothesis testing
- p-values
- Bayesian models
- the prior and posterior distribution
- the prior predictive distribution
- the *beta*-distribution
- examples 7.1.1,2,3 and 4

Problem 1

Give a formal proof that the two following definitions are equivalent:

- a probability measure Π on the set $\Theta \times S$
- a prior probability measure P on the set Θ as well as a statistical model $\Theta \Rightarrow S$

To this end, recall from class that given a probability measure on $\Theta \times S$, we can define a prior as $P(A) = \Pi(\Theta \times S)$ as well as a statistical model with density functions $f_\theta(x) = j(\theta, x)$ where $j(\theta, x)$ is the density function of Π. Conversely, given a statistical model with density functions $f_\theta(x)$ and a prior probability on Θ with density π, we consider $j(\theta, x) = \pi(\theta) f_\theta(x)$. These two construction then coincide.

Problem 2

Assume now that on S, we have observed the data s.
Show that the posterior density m is the density function of the measure given by $\Pi(A \times B|\Theta \times s)$ (as in exercise 1)

Problem 3

Practice your skills on Bayesian models by doing problems 7.1.1,2,7,9,11,12

Problem 4

Assume we consider the following Bayesian model: we let $\Theta = [0,1]$ with uniform distribution (ie the density function for the prior is simply $\pi(\theta) = 1$. We consider the set $S = \{0,1\}^n$ with binomial distribution.

- Show that the density function for the prior predictive m is given by $B(\alpha, \beta)$ where α is the number of 1's and β the number of 0's.
- Conclude that the posterior density function follows a distribution $\sim B(\alpha, \beta)$

Problem 5

Recall that by definition, the function $\beta(\alpha, \beta)$ is given by $\beta(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$.
Using integration on $\mathbb{R}^2$, we can prove that

$$\beta(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Use the properties you know from the gamma function to show that

- $\beta(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$
- $B(\alpha, \beta) = B(\alpha, \beta+1) + B(\alpha+1, \beta)$
- $B(\alpha+1, \beta) = B(\alpha, \beta) \cdot \frac{\alpha}{\alpha+\beta}$