DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES
UNIVERSITY OF TORONTO, Scarborough
MATC32: Graph Theory
Fall 2017

Name: ____________________

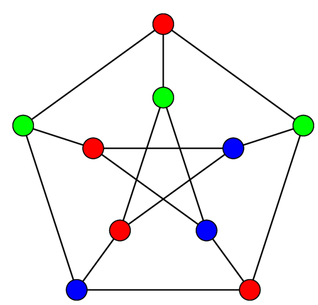
Student Nr.: ____________________

- No electronic aids
- No books, notebooks or scrap paper are permitted.
- Answer all questions in the booklet provided
- Show your work and justify your answers for full credit.

1. Recall the Petersen graph $G = (V, E)$: the vertices V are all sets of 2 elements $\{a, b\}$ for $a, b \in \{1, \ldots, 5\}$.

   (a) Draw this graph.

   Solution:

   (b) consider a vertex coloring $V \longrightarrow \{1, 2\}$. Show that there exist two adjacent vertices with the same color.

   Solution: As we know, a proper coloring is the same as partitioning the vertices into two independent subsets (those with color 1 and 2 resp.), in other words, a bipartition. Konig's theorem now guarantees the existence of such a subdivision if and only if the graph has no odd length cycle. However, we clearly see a cycle of length 5!

   Alternatively, you can also just pick a vertex and argue why you will always get stuck at some point in the construction of a 2-coloring by hand.

   (c) We now consider a vertex coloring $V \longrightarrow \{1, 2, 3\}$ through the following rule: the vertex $\{a, b\}$ has color 1 if the smallest element is 1, it has color 2 if the smallest element is 2 and 3 otherwise. Draw the colors on the graph.

Solution: This coloring is displayed on the graph above.

(d) recall that the chromatic number χ is the smallest amount of colors necessary such that there exists a coloring where no adjacent vertices have the same color. What is the chromatic number of Petersen graph?

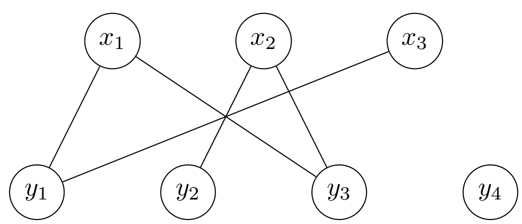
Solution: Point (b) demonstrates that this number is strictly larger that 2 and point (c) shows that the number is at most 3. The answer is thus 3.

2. Let M be an $m \times n$-matrix with entries in $\{0, 1\}$.
To M, we associate a graph $G = (V, E)$ as follows: consider vertex sets $V = \{x_1 \dots x_m\} \cup \{y_1, \dots y_n\}$ and add an edge $e \in E$ between vertices x_i and y_j if $M_{i,j} = 1$.

(a) Draw the graph associated to

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Solution:



(b) State Kónig's theorem for bipartite graphs:

Solution: In a bipartite graph, the size of a maximal matching set of edges coincides with the size of a minimal vertex cover. (alternatively, many people also answered: a graph is bipartite if and only if it has no odd length cycles, this is also correct)

(c) Let a *line* be either a row or column.
Use the above to show that the following two numbers coincide:

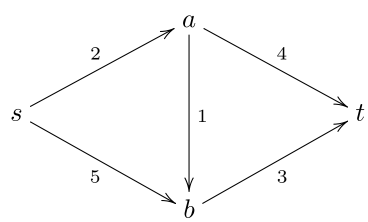
- the size of the smallest set of lines in which each entry 1 appears in some line
- the size of the largest set of 1's in the matrix no two of which appear on the same line.

Solution: Let us denote a line (i.e. row or column) by its corresponding vertex: row i corresponds to vertex x_i and column j corresponds to vertex y_j. The nonzero entries in a line then correspond to edges are adjacent to said vertex.
By the above correspondence, a choice S of lines that contains all 1's in the matrix now corresponds to choice of vertices S such that each edge in the graph is adjacent to some vertex in S: a vertex cover.
Similarly, a choice of 1's no two of which lie on the same line in turn corresponds to a choice of edges no two of which are adjacent to the same vertex: a matching.. The result now follows from Konig's theorem stated above

(d) find this number for the matrix in the above example

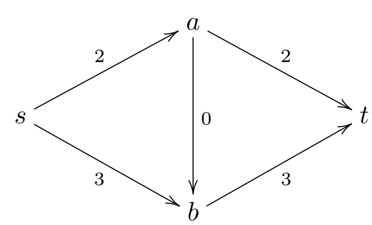
> **Solution:** We simply need to find a minimal vertex cover (or alternatively a maximal matching) in the above graph. Clearly $\{x_1, x_2, x_3\}$ is such a cover. The number is thus 3.

3. Consider the following network:



(a) perform the Ford-Fulkerton algorithm to find a maximum flow

(b) find a minimum cut

> **Solution:** This was discussed in class . The result of the algorithm is the following network:
>
> 
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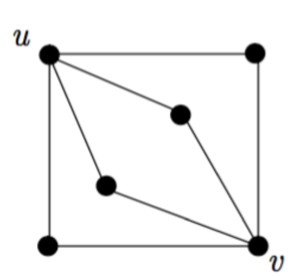
>
> As this clearly yields a flow whose residual graph is acyclic. The max-flow equals the min-cut with a value of 5. By letting $S = \{s, b\}$ and $T\{a, t\}$, we see that $||S, T|| = 5$

4. (a) State Menger's theorem:

> **Solution:** Let u, v be two distinct vertices. Then the minimal size of a edge cut for u, v coincides with the largest size of a set of edge-disjoint $u - v$ paths.
> Or: the minimal size of a vertex cut for two nonadjacent vertices u, v coincides with the largest size of internally disjoint $u - v$-paths.
> Or: the minimal size of an edge cut in a graph equals the maximal size of edge-disjoint paths in the graph.

(b) verify Menger's theorem on the following graph:



> **Solution:** Clearly, any u, v-path would have to go through one of the non-labeled vertices, and for each such vertex there is exactly one path. There are 4 edge-disjoint paths. Similarly, each of these 4 paths would have to be removed to disconnet u and v. This can be done by picking an edge for each path. The size of a minimal edge cut is thus 4 as well.

5. Assume $2m$ people participate in a chess tournament. They all play one opponent once over the course of $2m-1$ days. Let G be the bipartite graph with edge set $X \cup Y$ where X represents the $2m-1$ days and Y represents the $2m$ players. Connect a day with a player by an edge if the player won his game that day.

   (a) State the marriage condition for this problem.

   > **Solution:** The marriage condition states in general that for any subset $W \subset X$ and $\Gamma(W) = \{v \in Y | \exists u \in W : uv \in E(G)\}$, we have
   >
   > $$\Gamma(W) \leq W$$
   >
   > In this particular case, W is a set of days, and $\Gamma(W)$ is the set of players that are *adjacent* to the days in W. Since we connect players with days if they won on that day, the marriage condition simply states that during any period of k days, there are at least k players who won in that period.

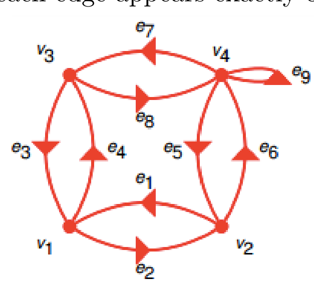
   (b) Argue (by contradiction) that the marriage condition is satisfied

   > **Solution:** Assume the condition isn't satisfied.
   > Then for some choice of k days, less that k people won a game during that period. In particular there is some person who lost all of his k games during that period..This means that k different people won however! A contradiction..

   (c) What can you conclude about the tournament? The graph that represents this tournament has a perfect matching. Ie: we can list a winning player for each day of the tournament without repeating players

6. Answer with true or false (and give an argument/counterexample when necessary)

   (a) the graph below has a path where each edge appears exactly once:



   > **Solution:** This is true: consider the path
   >
   > $$e_1, e_4, e_8, e_9, e_5, e_6, e_7, e_3, e_2$$

   (b) given a network with capacities $\in \mathbb{Z}$, the Ford-Fulkerton algorithm always finds a maximum flow.

Solution: This is true as well: indeed, the FF algorithm increases the value of the flow at each step. This value remains a positive integer. We now note that the value of any flow is always bounded by the value of a cut

(c) A cycle of length ≥ 3 becomes disconnected after removing 1 edge:

Solution: This is not true as it is known that any cycle remains connected after removing an edge by Konig's theorem

(d) A cycle of length ≥ 3 becomes disconnected after removing 2 edges.

Solution: This is true in the case where the two edges are not adjacent.