University of Toronto

Department of Computer & Mathematical Sciences

MATC32: Graph theory

Assignment Nr 1

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This week's list of problems is based off the material covered in weeks 1 through 3: i.e. covered in book sections 1.1, 1.2 and 4.3.
You should work on each individual problem and submit your solutions in the MATC32 dropbox by Friday evening of next week
You will be graded not only on your answers but on how you argue. So make sure you prove your arguments correctly or give counterexamples whenever necessary

Equivalence relations

Are the following relations equivalences? Either prove or disprove. In the former case, describe the classes:

- on $\mathbb{R}$, let $x \sim y \iff x - y \geq 0$.
- on a vector space V with subspace $W \subset V$: $x \sim y \iff x - y \in W$
- On a graph G, for two vertices we say $x \sim y$ iff they have the same number of adjacent edges

$\Longrightarrow$ Answer

- This is not an equivalence relation as it is not symmetric
- This is an equivalence relation. The classes are $\overline{x} = x + W$ or the subspace W, translated by the vector x
- This is also an equivalence relation where the classes are in correspondence with certain numbers (representing the degree of the vertex)

Graph invariants

Recall that a property is a graph invariant if two graphs share the same property whenever they are isomorphic. Give an example of a property you can associate to the data of a graph G which is **not** a graph invariant.

$\Longrightarrow$ Answer

There are many options, you could take the adjacency matrix of a graph for example.

Examples of graphs

Give an example of a graph satisfying the following properties, or argue why such an example can't exist:

- A connected graph G together with an vertex v such that $G \setminus v$ has exactly 4 connected components.
- a graph with an even number of vertices which forms a closed walk yet has contains no closed cycle.
- A network together with a non-maximal flow with a value of 6.

$\Longrightarrow$ Answer

- the graph with 5 vertices a, b, c, d, e and a single edge from e to vertices a, b, c, d is a counterexample.
- Two vertices with an edge between them form a counterexample
- take a graph with source s, sink t and middle vertices a, b, c, draw an edge from s to a, b, c and from a, b, c to t. endow each directed with flow s and capacity 100

Matching

Recall that in a bipartite graph $G = X \coprod Y$, a matching is a set of edges none of which are adjacent. A perfect matching is a matching where every vertex $x \in X$ is adjacent to some vertex $y \in Y$.

1. Do perfect matchings always exist?
2. Are perfect matchings necessarily unique?

$\Longrightarrow$ Answer

- a graph without edges can;t have a matching.
- No they are not. Take a bipartite graph with $X - \{a, b\}$ and $Y = \{c, d\}$ and define an edge for any choice of vertex between X and Y. This has two perfect matchings.

a Famous graph

We construct a graph G as follows: let $X = \{1, 2, 3, 4, 5\}$, let $V(G)$ be the set of all subsets of X of size two (for example $\{3, 4\} \in V(G)$). We join two vertices x and y by an edge iff the two subsets are disjoint.

1. Draw this graph
2. Is this graph bipartite?

$\Longrightarrow$ Answer

This graph is the Petersen graph which looks like a pentagram inside a pentagon. It is not bipartite as it is easy to find an odd cycle